

A Study on Fuzzy Algebra and Fuzzy Topology

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ABSTRACT: The theory of fuzzy sets was introduced by L. A. Zadeh in his classic paper “Fuzzy Sets” [1] in 1965. It enables us to describe mathematically the collection of objects for which there is no precise criteria for membership. For example, collection of beautiful women. But it is a fact that such classes of ill defined objects play an important role in certain real situations. In patten, classification fuzziness is a rule rather than an exception. Ever since its inception the theory of fuzzy sets has advanced in many disciplines. Theoretical advances have been made in many directions such as fuzzy Algebra, Fuzzy Analysis, Fuzzy Topology and Fuzzy logic. Today the theory of fuzzy sets is finding applications in a wide variety of fields such as computer science, artificial intelligence, decision analysis, Management science, and operations research and so on. Home appliances like washing machine, vacuum cleaner are made using fuzzy logic. My dissertation surveys some of the basic concepts and theories of “Fuzzy Algebra and Topology” which is a generalization of concepts to Fuzziness. We deal with Fuzzy subgroups of a group, level subgroups and homomorphism. We deals with Fuzzy direct product of fuzzy subgroups of subgroups. We describe the concepts of Fuzzy subrings and homomorphism of subrings and subfields. We describes about Fuzzy topological spaces, local countability, convergence, separability, product Fuzzy topology and fuzzy homeomorphism. We describe the concepts of Fuzzy subrings and homomorphism of subrings and subfields. We describes about Fuzzy topological spaces, local countability, convergence, separability, product Fuzzy topology and fuzzy homomorphism.

I. INTRODUCTION

The theory of fuzzy sets was introduced by L.A. Zadeh in his classic paper “Fuzzy Sets” [1] in 1965. It enables us to describe mathematically the collection of objects for which there is no precise criteria for membership. For example, collection of beautiful women. But it is a fact that such classes of ill defined objects play an important role in certain real situations. In patten, classification fuzziness is a rule rather than an exception.

Ever since its inception the theory of fuzzy sets has advanced in much discipline. Theoretical advances have been made in many directions such as fuzzy Algebra, Fuzzy Analysis, Fuzzy Topology and Fuzzy logic.

This dissertation surveys some of the basic concepts and theories of “Fuzzy Algebra and Topology” which is a generalization of concepts to Fuzziness.

II. FUZZY SUBGROUPS OF A GROUP

1.1 PRELIMINARY DEFINITIONS AND RESULTS

1.1.1 Definition

Let S be a set. A fuzzy subset A of S is a mapping $A : S \rightarrow [0,1]$, $A(x)$ is called the grade of membership of x in A . The degree of belonging increases if $A(x)$ is nearer to 1.

1.1.2 Definition

Let A and B be fuzzy subsets of a Set S .

Then (1) The union of A and B is denoted by $A \cup B$ and is defined as $(A \cup B)(x) = \text{Max} \{A(x), B(x)\} \forall x \in S$.

The intersection of A and B is denoted by $A \cap B$ and is defined as $(A \cap B)(x) = \text{Min} \{A(x), B(x)\} \forall x \in S$.

Complement of A is \bar{A} and is defined as $\bar{A}(x) = 1 - A(x) = 1 - A(x) \forall x \in S$.

$A \subseteq B$ if $A(x) \leq B(x) \forall x \in S$.

$A = B$ if $A(x) = B(x) \forall x \in S$.

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If $\{A_i\}_{i \in I}$ is a collection of fuzzy subsets of S then their union is defined as $(\cup_{i \in I} A_i)(x) = \text{Sup} \{A_i(x)\} \forall x \in S$ and their intersection is defined as $(\cap_{i \in I} A_i)(x) = \text{Inf} \{A_i(x)\} \forall x \in S$.

1.2 HOMOMORPHISM

1.2.1 Definition

Let A be a fuzzy subset of S and f is a function defined on S, then the fuzzy set B in f(s) defined by $B(y) = \text{Sup}_{x \in f^{-1}y} A(x) \forall y \in f(s)$ is called the image of A under f. The image of A is also denoted by f(A)

$$(ie) B = f(A).$$

1.2.1 Theorem

A homomorphic image of a fuzzy subgroup is a fuzzy subgroup if the sup property holds. Further, the pre image of a fuzzy subgroup is a fuzzy subgroup.

Proof

Let G be a group and let f be a homomorphism on G.

Let A be a fuzzy subgroup of G. Then its image $f(A) = B$ is a fuzzy subset of f(G).

$$\text{Then } B(y) = \text{Sup}_{x \in f^{-1}y} A(x) \forall y \in f(G)$$

We prove B is a fuzzy subgroup of f(G). Let $f(x), f(y) \in f(G)$ and let $x_0 \in f^{-1}(f(x))$ and $y_0 \in f^{-1}(f(y))$ such that

$$A(x_0) = \text{Sup}_{t \in f^{-1}(f(x))} A(t) \quad A(y_0) = \text{Sup}_{t \in f^{-1}(f(y))} A(t) \text{ Since A has sup property.}$$

Now,

$$\begin{aligned} B(f(x) f(y)) &= \text{Sup}_{z \in f^{-1}(f(x)f(y))} A(z) \\ &\geq \min(A(x_0), A(y_0)) \\ &= \min\{B(f(x)), B(f(y))\} \end{aligned}$$

And

$$\begin{aligned} B((f(x))^{-1}) &= \text{Sup}_{z \in f^{-1}((f(x))^{-1})} A(z) \\ A(x_0^{-1}) &\geq A(x_0) = B(f(x)) \end{aligned}$$

Hence B is a fuzzy subgroup of f(G). Thus homomorphism image of a fuzzy subgroup is a fuzzy subgroup.

1.2.2 Definition

Let f be a function defined on Set S. If B is a fuzzy subset of f(s) then the fuzzy set $A = B \circ f$ in s defined by $A(x) = B(f(x)) \forall x \in S$ is called preimage of B is also denoted by $f^{-1}(B)$.

$$(ie) A = f^{-1}(B)$$

1.2.3 Definition

A fuzzy subset A of S is said to have the sup property if for any subset $T \subset S$, there exist $t_0 \in T$ such that

$$A(t_0) = \text{Sup}_{t \in T} A(t)$$

III. FUZZY DIRECT PRODUCT OF FUZZY SUBGROUPS OF SUBGROUPS

2.1 PRELIMINARY DEFINITIONS AND THEOREMS:

2.1.1 Definition

Let A, B be fuzzy subgroups of a group G we say that A is equivalent to B ($A \sim B$) if $FA = FB$.

Obviously " \sim " in an equivalence relation on the set of all fuzzy subgroup of G.

2.1.1 Theorem

If A and B are fuzzy subgroups of a group G, then $A \sim B$ if and only if $\text{Im } A = \text{Im } B$ and $FA = FB$ where $FA = \{At : t \in \text{Im } A\}$.

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2.1.2 Definition

For a fuzzy subgroup A of G , we associate subsets A_* and A^* of G defined by $A_* = A_{A(e)}$ and $A^* = \{x \in G : A(x) \neq A(e)\}$

2.1.2 Theorem

Let A be fuzzy subset of a group G . Then A is a fuzzy subgroup of G if and only if each level subset A_t , t is an element of $\text{Im } A$, is a subgroup of G .

Proof

Let A be a fuzzy subgroup of a group G and $\text{Im } A = \{t_0, t_1 \dots t_n\}$
Such that $t_0 > t_1 > \dots > t_n$ then the level subgroups form a chain

$$G = A_{t_n}, A_{t_{n-1}} \dots A_{t_0}$$

Also for a finite chain

$$G = G_n > G_{n-1} > \dots > G_0$$

of subgroup of G , there exist a fuzzy subgroup A of G whose level subgroups are precisely the members of the given chain. Actually the fuzzy subgroups which have this chain as their chain of level subgroups are the elements of the equivalence class given by definition.

Therefore we have $A : G = G_{t_0} > G_{t_1} > \dots > G_{t_n}$ to mean that A is a fuzzy subgroup of G with $\text{Im } A = \{t_0, t_1 \dots t_n\}$ such that $t_0 > t_1 > \dots > t_n$ and $G_{t_n}, G_{t_{n-1}} \dots G_{t_0}$ are the level subgroups of A .

2.1.3 Definition

A fuzzy subgroup A of a group G is called fuzzy subnormal subgroup of G if the chain of level subgroups of A forms a subnormal series of G .

$$(ie.) A:G = G_{t_0} > G_{t_1} > \dots > G_{t_n}$$

It is obvious that every fuzzy normal subgroup is a fuzzy subnormal subgroup however the converse is not always true.

2.1.3 Theorem

A fuzzy subgroup A of a group G having the chain

$A:G = G_{t_0} > G_{t_1} > \dots > G_{t_n}$ is a fuzzy normal subgroup of G if and only if the chain of level subgroups of A forms a normal series of G .

$$(ie.) (G_{t_i} < G \forall 1 \leq i \leq n).$$

2.1.4 Definition

Let A be a fuzzy subgroup of G . The number $|\text{Im } A|$ is called the length of A .

2.1.4 Theorem

Suppose G has a principal series and let H, K be normal subgroups of G with $K < H$. Then H/K is a principal factor of G if and only if H/K is a minimal normal subgroup of G/K .

2.1.5 Definition

A fuzzy normal subgroup A of G is called a fuzzy subgroup of principal length if the chain of level subgroups of A forms a principal (chief) series ((ie) each G_{t_i} is maximal subgroup of $G_{t_{i-1}}$).

2.1.5 Theorem

A group G satisfies the chain conditions (ascending and descending) if and only if every fuzzy subgroup of G is of finite length.

It is known from group theory that a group has a composition (principal) series if and only if it satisfies (normal) chain conditions. Thus translating this to the fuzzy setting and using the fact that all subnormal (normal) chain of such groups have equivalent composition (principal) series as refinements.

2.1.6 Definition

A fuzzy subnormal subgroup A of G is called a fuzzy subgroup of composition length if the chain of level subgroup of A forms a composition series of G .

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2.1.6 Theorem

A group G has composition (principal) series if and only if every fuzzy subnormal (normal) subgroup of G is of finite length.

IV. FUZZY SUBRINGS

3.1 Preliminary Definitions and Results

3.1.1 Definition

Let R be a ring. A fuzzy subset A of the ring R is called a fuzzy subring of R if for all $x, y \in R$,
 $A(x-y) \geq \min \{A(x), A(y)\}$
 $A(xy) \geq \min \{A(x), A(y)\}$

3.1.1 Theorem

If $\{A_n; n \in \mathbb{Z}_+\}$ is a collection of fuzzy ideals of a ring R such that
 $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq \dots$ then, $\bigcup_{n \in \mathbb{Z}_+} A_n$ is a fuzzy ideal of R. (where A_n denotes a fuzzy ideal of a ring R. This notation should not be mixed up with that of a level subset (level ideal).

Proof

Let $x, y \in R$. For $i, j \in \mathbb{Z}_+$ there exists $k \in \mathbb{Z}_+$ such that $A_i(x) \leq A_k(x)$ and $A_j(y) \leq A_k(y)$ and so.

$$\min \{A_i(x), A_j(y)\} \leq \min \{A_k(x), A_k(y)\}$$

Since A_k is a fuzzy ideal of R,

$$A_k(x-y) \geq \min \{A_k(x), A_k(y)\}$$

$$\therefore \min \{A_i(x), A_j(y)\} \leq A_k(x-y)$$

Now,

$$\begin{aligned} \min \left\{ \bigcup_{n \in \mathbb{Z}_+} A_n(x), \bigcup_{n \in \mathbb{Z}_+} A_n(y) \right\} &= \min \left\{ \sup_{n \in \mathbb{Z}_+} A_n(x), \sup_{n \in \mathbb{Z}_+} A_n(y) \right\} \\ &= \sup_{i, j \in \mathbb{Z}_+} \{ \min(A_i(x), A_j(y)) \} \\ &= \sup_{k \in \mathbb{Z}_+} A_k(x-y) = \left(\bigcup_{n \in \mathbb{Z}_+} A_n \right)(x-y) \end{aligned}$$

$$\text{Finally, } \left(\bigcup_{n \in \mathbb{Z}_+} A_n \right)(xy) \geq A_i(xy)$$

$$\geq \max \{A_i(x), A_i(y)\} \quad \forall i \in \mathbb{Z}_+$$

$$\text{This } \left(\bigcup_{n \in \mathbb{Z}_+} A_n \right)(xy) \geq \max \left\{ \left(\bigcup_{n \in \mathbb{Z}_+} A_n \right)(x), \left(\bigcup_{n \in \mathbb{Z}_+} A_n \right)(y) \right\}$$

Hence $\left(\bigcup_{n \in \mathbb{Z}_+} A_n \right)$ is a fuzzy ideal of R.

3.1.2 Theorem

Let A be a fuzzy ideal of a ring R, then the map $f: R \rightarrow R_A$ defined by $f(x) = A_x^* \quad \forall x \in R$ is a homomorphism. With kernel A_t where $t = A(0)$.

$$\text{Further } \frac{R}{A_t} \approx R_A.$$

Proof

Define $f: R \rightarrow R_A$ by $f(x) = A_x^*$ for all $x \in R$

Since A is a fuzzy ideal of R, we have $A(x) = A(0) \quad \forall x \in R$

By above Lemma, $A(x) = A(0) \Leftrightarrow A_x^* = A_0^*$.

$$\begin{aligned} \text{Now } f(x+y) &= A_{x+y}^* \\ &= A_x^* + A_y^* \end{aligned}$$

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$$\begin{aligned} \text{and } f(xy) &= A_{xy}^* \\ &= f(x) + f(y) \\ &= A_x^* A_y^* \\ &= f(x) f(y) \end{aligned}$$

∴ f is a homomorphism

Clearly, f is onto.

$$\begin{aligned} \text{Now, Ker } f &= \{x \in R \mid f(x) = A_0^*\} \\ &= \{x \in R \mid A_x^* = A_0^*\} \\ &= \{x \in R \mid A(x) = A(0)\} \\ &= \{x \in R \mid A(x) = t\}, t = A(0) \\ &= A_t \end{aligned}$$

By fundamental theorem of ring homomorphism

$$\therefore \frac{R}{A_t} \approx R_A.$$

Know the homomorphism image of a Noetherian ring is Noetherian and the homomorphic image of an Artinian ring is Artinian.

Corollary

If A is any fuzzy ideal of a ring R and R is Noetherian (Artinian) then R_A is Noetherian (Artinian).

V. FUZZY TOPOLOGICAL SPACES

4.1 TOPOLOGICAL SPACES

4.1.1 Definition

A fuzzy topology is a family T of fuzzy sets in X which satisfies the conditions

$\phi, X \in T$

If $A, B \in T$ then $A \cap B \in T$

If $A_i \in T$ for each $i \in I$, then $\cup A_i \in T$

4.1.1 Theorem

A fuzzy set A is open if and only if for each fuzzy set B contained in A, A is a neighborhood of B.

Proof

If A is open, obviously, there exist for each fuzzy set $B \subset A$, A is neighborhood.

Conversely, Let A be a neighborhood of each $B \subset A$. Since $A \subset A$, there exist an open fuzzy set O such that $A \subset O \subset A$.

4.1.2 Definition

A fuzzy set U in a fuzzy topological space (X, T) is a neighborhood, (or neighborhood for a short), of a fuzzy set A if and only if there exists an open fuzzy set O such that $A \subset O \subset U$.

4.1.2 Theorem

If U is the neighborhood system of a fuzzy set, then finite intersection of members of U belong to U and each fuzzy set which contains a member of U belongs to U.

Proof

If R and S are neighborhoods of a fuzzy set A, there are open neighborhoods R_0 and S_0 contained in R and S respectively.

Then $R \cap S$ contains the open neighborhood $R_0 \cap S_0$ and is hence a neighborhood of A. Thus intersection of two members of U is a member of U, which is true for any finite members of members of U.

Hence, if a fuzzy set R contains a neighborhood of A it contains an open neighborhood of A and consequently is itself a neighborhood.

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4.1.3 Definition

A sequence of fuzzy sets, say $\{A_n, n=1, 2, \dots\}$ is eventually contained in a fuzzy set A if and only if there is an integer m such that, if $n \geq m$ then $A_n \subset A$. The sequences frequently contained in A if and only if for each integer m there is an integer n such that $n \geq m$ and $A_n \subset A$. If the sequence is in a fuzzy topological space (X, T) . Then we say that the sequence converges to a fuzzy set A if and only if it is eventually contained in each neighborhood of A .

4.1.4 Definition

Let N be a map from the set of non negative integers to the set of a non-negative integer. Then the sequence $\{B_i, i = 1, 2, \dots\}$ is a subsequence of a sequence $\{A_n, n=1, 2, \dots\}$ if and only if there is a map N such that $B_i = A_{N(i)}$ and for each integer m there is an integer n such that $N(i) \geq m$ whenever $i \geq n$

4.2 COMPACTNESS

Let E be a set and $I = [0, 1]$ and $\delta \subset I^E$ is a fuzzy topology on E if and only if (i) $\forall \alpha$ constant $\alpha \in \delta$ (ii) $\forall \mu, r \in \delta \mu \wedge r \in \delta$ (iii) $\forall (\mu_i)_{i \in I}, r \in \delta \mu \wedge r \in \delta$. We give below another form of compactness.

4.2.1 Definition

The fuzzy topological space (or quasi fuzzy topological space) (E, δ) is fuzzy compact if and only if each constant fuzzy set in (E, δ) is fuzzy compact.

4.2.1 Theorem

The fuzzy topological space $(E, w(\mathfrak{T}))$ is compact, if and only if the space (E, \mathfrak{T}) is compact, when \mathfrak{T} is a fuzzy topology for X .

Proof

Let $\beta \subset w(\mathfrak{T})$ be such that

$\sup_{\mu \in \beta} \mu \geq \alpha > 0$ and ε such that $\alpha > \varepsilon > 0, \forall \mu \in \beta$ put $\mu^\varepsilon = \mu + \varepsilon [0, \alpha] = I_\alpha$

Then $\forall \mu \in \beta, L(\mu^\varepsilon) = \{(x, r) : \mu^\varepsilon(x) > r\}$ is an open set in $E \times R$ and

$$\bigcup_{\mu \in \beta} L(\mu^\varepsilon) \supset E \times I_\alpha$$

Since $E \times I_\alpha$ is compact there exist a finite sub family $\beta_0 \subset \beta$ such that

$$\bigcup_{\mu \in \beta} L(\mu^\varepsilon) \supset E \times I_\alpha \text{ and then clearly, } \sup_{A \in \beta_0} \mu \geq \alpha - \varepsilon.$$

Conversely,

Suppose $B \in \mathfrak{T}$ is an open cover of E then $\sup_{A \in B} \chi_A = 1$.

Where χ_A is the characteristic function of A .

Choose $\varepsilon \in [0, 1]$ then it follows from the fuzzy compactness of

$[E, w(\mathfrak{T})]$ that there exist a finite subset $B_0 \supset B$ such that $\sup_{A \in B_0} \chi_A \geq 1 - \varepsilon$.

It is clear that B_0 is a finite subcover of B .

4.2.2 Definition

The fuzzy topological space (E, δ) is weakly fuzzy compact if and only if I_E is fuzzy compact.

VI. CONCLUSION

My dissertation surveys some of the basic concepts and theories of "Fuzzy Algebra and Topology" which is a generalization of concepts to Fuzziness. We deal with Fuzzy subgroups of a group, level subgroups and homomorphism. We deals with Fuzzy direct product of fuzzy subgroups of subgroups. We describe the concepts of Fuzzy subrings and homomorphism of subrings and subfields. We describes about Fuzzy topological spaces, local countability, convergence, separability, product Fuzzy topology and fuzzy homeomorphism. We describe the concepts of Fuzzy subrings and homomorphism of subrings and subfields. We describes about Fuzzy topological spaces, local countability, convergence, separability, product Fuzzy topology and fuzzy homomorphism.

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